An Application of Goal-oriented Error Estimation to Shock Loaded Elastomeric Materials

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INTRODUCTION

Background: Methods for *a posteriori* error estimation of finite element approximations have been primarily developed for linear, elliptic (steady state) problems. (some exceptions. Verfurth, Becker & Rannacher, Oden and Prudhomme, Stein et al.)

Goal: To develop goal-oriented *A Posteriori* error estimates for finite element approximations of a class of highly nonlinear problems in continuum mechanics.

- highly-nonlinear dynamical behavior of elasto-plastic, rate-sensitive structures subjected to shock loading and contact
OUTLINE

1. Brief review of goal-oriented error estimation
2. Nonlinear mechanics in goal-oriented framework
   - Updated Lagrangian variational formulation
   - Develop dual variational formulation for a meaningful quantity of interest
3. Numerical Results
4. Conclusions
GOAL-ORIENTED ERROR ESTIMATION∗

Abstract Primal Problem

Find \( u \in \mathcal{V} \) such that
\[
B(u; v) = F(v) \quad \forall v \in \mathcal{V}
\]

Compute the error in a quantity of interest \( Q(u) : \mathcal{V} \rightarrow \mathbb{R} \)

The associated dual problem: Find \( p \in \mathcal{V} \) such that
\[
B'(u; w, p) = Q'(u; w) \quad \forall w \in \mathcal{V}
\]

\[
Q'(u; v) \equiv \lim_{\theta \to 0} \frac{1}{\theta} [Q(u + \theta v) - Q(u)]
\]

\[
B'(u; v, p) \equiv \lim_{\theta \to 0} \frac{1}{\theta} [B(u + \theta v; p) - B(u; p)]
\]

GOAL-ORIENTED ERROR ESTIMATION*

Let \( u_h \in V_h \subset V \) and \( \tilde{p} \in \tilde{V}, V_h \subset \tilde{V} \subset V \). Then

\[
\mathcal{E} = Q(u) - Q(u_h) = \mathcal{R}(u_h; \tilde{p}) + \Delta(e, \varepsilon)
\]

\( \mathcal{R}(u_h; v) = F(v) - B(u_h; v) = \text{the residual functional} \)

\( e = u - u_h, \quad \varepsilon = p - \tilde{p} = \text{error components} \)

\( \Delta(e, \varepsilon) = \text{remainder (H.O.T. in } e, \varepsilon) \)
PRIMAL PROBLEM: Governing Equations

Nonlinear Continuum Mechanics

\[ \rho J = \rho_o \]
\[ \rho \frac{d\nabla}{dt} = \nabla_x \cdot \sigma \]
\[ \frac{d\mathbf{u}}{dt} = \mathbf{v} \]
\[ \frac{d}{dt} (\rho e) = \sigma : \mathbf{D} \]

Definitions

\( u = \) displacement  \( J = \det(I + \nabla_x u) \)  \( \sigma = \) Cauchy stress
\( D = \) rate of deformation  \( \mathbf{v} = \) velocity  \( e = \) internal energy
PRIMAL PROBLEM: Weak Formulation

Find \((u, v) \in \mathcal{V} (= \mathcal{U} \times \mathcal{W})\) such that:

\[ B((u, v); (z, w)) = F((z, w)) \quad \forall (z, w) \in \mathcal{V} \]

\[
B((u, v); (z, w)) =
\int_0^T \int_{\Omega_t} \left( \frac{d}{dt} v \cdot w + \sigma : \nabla_x w + \frac{d}{dt} u \cdot z - \rho v \cdot z \right) dx dt
\]

\[
+ \int_{\Omega_o} \left( \rho_o v(X, 0) \cdot w(X, 0) + \rho_o u(X, 0) \cdot z(X, 0) \right) dX
\]

\[
F((z, w)) =
\int_0^T \int_{\Gamma_t^N} g \cdot w dA + \int_{\Omega_o} \left( \rho_o v_o(X) \cdot w(X, 0) \right) dX
\]

\[
+ \int_{\Omega_o} \left( \rho_o u_o(X) \cdot z(X, 0) \right) dX
\]
DUAL PROBLEM

Example of a quantity of interest:

\[ Q((u, v)) = \int_{\Omega_T} K(x)u_z \, dx = \int_{\omega_T} u_z \, dx \]

Dual Problem:

Find \((p, q) \in \mathcal{V}\) s.t.

\[ B'(((u, v); (z, w), (p, q)) = Q'((u, v); (z, w)) \quad \forall(z, w) \in \mathcal{V} \]

\((p, q)\) are the influence functions corresponding to \((u, v)\)
DUAL PROBLEM

\[B'( (u, v); (z, w), (p, q) ) = \int_0^T \int_{\Omega_t} (\rho \frac{dw}{dt} \cdot q + \rho w \cdot p + \rho \frac{dz}{dt} \cdot p) \, dx dt\]

\[\quad + \int_0^T \int_{\Omega_t} (\nabla_x q : \delta \sigma^u : \nabla_x z + \nabla_x p : \delta \sigma^v : \nabla_x w) \, dx dt\]

\[\quad + \int_{\Omega_o} (\rho_o w(\mathbf{X}, 0) \cdot q(\mathbf{X}, 0) + \rho_o z(\mathbf{X}, 0) \cdot p(\mathbf{X}, 0)) \, dX\]

\[= \int_0^T \int_{\Omega_t} (-\rho w \cdot \frac{dq}{dt} - \rho w \cdot p - \rho z \cdot \frac{dp}{dt}) \, dx dt\]

\[\quad + \int_0^T \int_{\Omega_t} (\nabla_x q : \delta \sigma^u : \nabla_x z + \nabla_x p : \delta \sigma^v : \nabla_x w) \, dx dt\]

\[\quad + \int_{\Omega_T} (\rho w(\mathbf{x}, T) \cdot q(\mathbf{x}, T) + \rho z(\mathbf{x}, T) \cdot p(\mathbf{x}, T)) \, dx\]
DUAL PROBLEM

Taylor series for stress variation

\[ \sigma_{ij}(\nabla_x u + \theta \nabla_x z, \nabla_x v + \theta \nabla_x w) = \sigma_{ij}(\nabla_x u, \nabla_x v) \]
\[ + \theta \frac{\partial \sigma_{ij}}{\partial (u_{m,n})} z_{m,n} + \theta \frac{\partial \sigma_{ij}}{\partial (v_{m,n})} w_{m,n} + O(\theta^2) \]

Derivative of quantity of interest

\[
\delta \sigma^u_{ijmn} = \frac{\partial \sigma_{ij}}{\partial (u_{m,n})} \quad \delta \sigma^v_{ijmn} = \frac{\partial \sigma_{ij}}{\partial (v_{m,n})}
\]

\[ Q^\prime((u, v); (z, w)) = \int_{\Omega_T} K(x) z_z(x) dx \]
TARGET PROBLEMS - NUMERICAL EXAMPLES

- Lagrangian mesh
- Shock loading
- Steel plate
- Elastomeric lining

![Diagram showing pressure loading, steel, and elastomer layers with a graph of pressure loading history.](image)
CONSTITUTIVE EQUATIONS - Primal Problem

Shock loading conditions indicate need for high-strain-rate constitutive equation

Steel:
- Assume hypoelastic - plastic model for stress state
- Use Mie-Gruneisen model for equation of state
- Use Johnson-Cook model for yield stress

Elastomer:
- Assume hydrodynamic stress state
- Use Mie-Gruneisen model for equation of state
CONSTITUTIVE EQUATIONS - Dual Problem

- Dual formulation restricted to elastomer
- Hydrodynamic state of stress for elastomer

\[ \sigma_{ij} = -p\delta_{ij} \]

- Energy dependence of pressure is assumed negligible
- The dependence on \( \nabla_x v \) and histories are ignored

\[ \delta\sigma^v_{ijmn} = \frac{\partial p}{\partial (v_{m,n})} \delta_{ij} = 0 \]

\[ \delta\sigma^u_{ijmn} = \frac{\partial p}{\partial (u_{m,n})} \delta_{ij} = \frac{\partial p}{\partial \rho} \cdot \frac{\rho_o}{J^2} \cdot C_{mn} \delta_{ij} \]
CONSTITUTIVE EQUATIONS - Dual Problem

\[ C_{11} = (F_{11} \ast F_{22} \ast u_{3,3} \ast F_{33} - F_{21} \ast F_{12} \ast u_{3,3} \ast F_{33} + F_{11} \ast F_{22} - F_{21} \ast F_{12}) \ast u_{2,2} + F_{11} + F_{11} \ast u_{3,3} \ast F_{33} \]
\[ C_{22} = (F_{11} \ast F_{22} \ast u_{3,3} \ast F_{33} - F_{21} \ast F_{12} \ast u_{3,3} \ast F_{33} + F_{11} \ast F_{22} - F_{21} \ast F_{12}) \ast u_{1,1} + F_{22} \ast u_{3,3} \ast F_{33} + F_{22} \]
\[ C_{33} = (F_{11} \ast u_{2,2} \ast F_{22} \ast F_{33} - u_{2,2} \ast F_{21} \ast F_{12} \ast F_{33} + F_{11} \ast F_{33}) \ast u_{1,1} + u_{2,1} \ast F_{12} \ast F_{33} + u_{2,2} \ast F_{22} \ast F_{33} + u_{1,2} \ast F_{21} \ast u_{2,1} \ast F_{12} \ast F_{33} - u_{2,1} \ast F_{11} \ast u_{1,2} \ast F_{22} \ast F_{33} + u_{1,2} \ast F_{21} \ast F_{33} + F_{33} \]
\[ C_{12} = (F_{21} \ast F_{12} \ast u_{3,3} \ast F_{33} - F_{11} \ast F_{22} \ast u_{3,3} \ast F_{33} + F_{21} \ast F_{12} - F_{11} \ast F_{22}) \ast u_{2,1} + F_{21} + F_{21} \ast u_{3,3} \ast F_{33} \]
\[ C_{21} = (F_{21} \ast F_{12} \ast u_{3,3} \ast F_{33} - F_{11} \ast F_{22} \ast u_{3,3} \ast F_{33} + F_{21} \ast F_{12} - F_{11} \ast F_{22}) \ast u_{1,2} + F_{12} \ast u_{3,3} \ast F_{33} + F_{12} \]
\[ C_{13} = 0 \quad C_{31} = 0 \quad C_{23} = 0 \quad C_{32} = 0 \]
NUMERICAL SOLUTION OF PRIMAL PROBLEM

- Explicit time stepping
- Bilinear shape functions
- Lumped consistent mass matrix
- Jaumann-rate based stress update
RESULTS FOR PRIMAL PROBLEM

Meshes: Uniform mesh in region of highest intensity shock loading

- Four elements across thickness of elastomer
- Seven elements across thickness of elastomer
- Ten elements across thickness of elastomer
- Thirteen elements across thickness of elastomer
RESULTS FOR PRIMAL PROBLEM

Steel on top: Sliding boundary condition

Animation Linux  Animation Windows
RESULTS FOR PRIMAL PROBLEM

Elastomer on top: Perfectly bonded boundary condition

Animation Linux  Animation Windows
NUMERICAL SOLUTION OF DUAL PROBLEM

- Implicit time stepping
  - Data from primal problem was saved
  - Mesh is pre-determined from the primal problem
- Consistent mass matrix
- Quadratic shape functions
- Contact algorithm included in formulation
RESULTS FOR DUAL PROBLEM

Steel top: Sliding boundary condition (axial component of dual solution)

Animation

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Animation Windows
RESULTS FOR DUAL PROBLEM

Steel top: Sliding boundary condition (radial component of dual solution)

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ERROR ESTIMATE CALCULATIONS

Let \((p^{h_2}, q^{h_2})\) be the quadratic approximations of \((p, q)\)

Then the error is estimated as:

\[
Q((u, v)) - Q((u^h, v^h)) 
\approx R((u^h, v^h); (p, q)) 
\approx R((u^h, v^h); (p^{h_2}, q^{h_2}))
\]

- Residuals identically zero for bilinear influence functions
- Primal solution projected into higher dimension dual solution space
ERROR ESTIMATE CALCULATIONS

temporal contribution to error

Error in the quantity of interest versus time
ERROR ESTIMATE CALCULATIONS

final error estimates

Error = \( Q((u, v)) - Q((u_h, v_h)) \)
CONCLUDING REMARKS

- Theory of goal-oriented error estimation has been applied to a class of highly nonlinear shock loaded problems
- Converging error estimates for a meaningful quantity of interest are demonstrated
- Extensions of dual problem to wider range of constitutive equations
  - plasticity
  - visco-elasticity
- Dual formulation for ALE/Fluid-Structure interaction
- Goal oriented adaptivity